$W_2$-RECURRENT LP-SASAKIAN MANIFOLD

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Abstract

In this paper, we study the properties of the $W_2$-recurrent LP-Sasakian manifold. We prove symmetric and skew-symmetric properties of the $W_2$-curvature tensor.

1. Introduction

In this paper, we shall study the $W_2$-recurrent LP-Sasakian manifold. An $n$-dimensional real differentiable manifold $M_n$ is said to be Lorentzian Para (LP)-Sasakian manifold if it admits a $(1,1)$ tensor field $F$, a $C^\infty$ vector field $T$, a $C^\infty$ 1-form $\alpha$ and a Lorentzian metric $g$ which satisfy (Mishra [2]):

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\[ A(T) = -1, \]  
\[ \overline{X} = X + A(X)T, \]  
\[ g(\overline{X}, \overline{Y}) = g(X, Y) + A(X)A(Y), \]  
\[ g(X, T) = A(X), \ D_X T = \overline{X}, \]  
\[ (D_X F)(Y) = \{g(X, Y) + A(X)A(Y)\}T + \{X + A(X)\}A(Y), \]  

where \( \overline{X} = F(X) \).

In an LP-Sasakian manifold \( M_n \) with structure \( (F, T, A, g) \), it can be seen that (Pokhariyal [5]):

\[ \overline{T} = 0, \ A(X) = 0, \]  
\[ \text{rank}(F) = n - 1. \]  

Moreover, if we put

\[ 'F(X, Y) = g(\overline{X}, Y), \]  
then the tensor \( 'F(X, Y) \) is symmetric in \( X \) and \( Y \).

In an \( n \)-dimensional LP-Sasakian manifold with the structure \( (F, T, A, g) \), we have the Riemannian curvature tensor as:

\[ R(X, Y, Z, T) = g(Y, Z)A(X) - g(X, Z)A(Y), \]  

where \( g(X, Z) \) is the metric tensor representing potential and

\[ Ric(X, Y) = g(QX, Y), \]  
is the Ricci tensor representing the matter tensor. Pokhariyal and Mishra [3] have defined a tensor

\[ 'W_2(X, Y, Z, U) \]  
\[ = 'R(X, Y, Z, U) + \frac{1}{(n - 1)}[g(X, Z)Ric(Y, U) - g(Y, Z)Ric(X, U)] \]  

to study its physical and geometric properties. This tensor is skew-symmetric.
in $X$ and $Y$. Breaking this tensor into skew-symmetric and symmetric parts, on contraction vanishes in an Einstein space. This allows Pirani formulation of gravitational waves to the Einstein space with the help of skew-symmetric part (Pokhariyal [7]). However, $W_2$ does not satisfy the cyclic property.

2. $W_2$-LP Sasakian Manifold

In this section, we study some of the geometrical properties of $W_2$-curvature tensor which is recurrent on LP-Sasakian manifold $M_n$ in a Sasakian manifold.

If we consider an LP-Sasakian manifold $M_n$ which is $W_2$-recurrent, then we have (Pokhariyal [4])

$$(D_U W_2)(X, Y)Z = B(U)W_2(X, Y)Z,$$  

(2.1)

where $B$ is a non-zero 1-form and $W_2$ curvature tensor is given by

$$W_2(X, Y)Z = R(X, Y)Z + \frac{1}{(n-1)}[g(X, Y)QY - g(Y, Z)QX].$$  

(2.2)

It is noted that $Q$ is the symmetric endomorphism of the tangent space at each point to the Ricci tensor

$$QX = (n-1)X.$$  

(2.3)

It is shown by De and Guha [1] that if in a Riemannian manifold (2.1) holds, then

$$R(X, Y)W_2(Z, U)V = 0,$$  

(2.4)

where $R(X, Y)Z$ is simply the derivation of the tensor algebra at each point of $M_n$ for its tangent vectors.

Using (1.4), we have

$$W_2(X, Y, Z, T) = g(W_2(X, Y)Z, T) = A(W_2(X, Y)Z).$$  

(2.5)
Further use of (2.2) gives

\[ A(W_2(X, Y)Z) = g(Y, Z)A(X) - g(X, Z)A(Y) \]
\[ + \frac{1}{(n-1)} [g(X, Z)g(QY, T) - g(Y, Z)g(QX, T)] \]
\[ = g(Y, Z)A(X) - g(X, Z)A(Y) \]
\[ + \frac{1}{(n-1)} [(n-1)A(Y)g(X, Z) - (n-1)A(X)g(Y, Z)] \]
\[ = g(Y, Z)A(X) - g(X, Z)A(Y) \]
\[ + g(X, Z)A(Y) - g(Y, Z)A(X) \]
\[ = 0. \quad (2.6) \]

This result holds for all vector fields \( X, Y, Z \).

**Theorem 2.1.** The LP-Sasakian manifold which is \( W_2 \)-reCURRENT with \( R(X, Y)W_2(Z, U)V = 0 \), is a \( W_2 \)-symmetric manifold.

**Proof.** From the above equation (2.6),

\[ - W_2(Z, R(X, Y)U)V - W_2(Z, U)R(X, Y)V \]
\[ = 0. \quad (2.7) \]

Using (1.4), we get

\[ g(R(T, Y)W_2(Z, U)V, T) - g(W_2(Z, U)R(T, Y)V, T) = 0, \]
\[ -g(W_2(Z, R(T, Y)UV, T) - g(W_2(Z, U)R(T, Y)V, T) = 0. \]

Expanding the terms and using (2.4), we get

\[ ^{1}R(T, Y, W_2(Z, U)V, T) \]
\[ = A(T)W_2(Z, U, Y) - A(T)A(W_2(Z, U)V) = -W_2(Z, U, V, Y). \quad (2.8) \]
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$\mathcal{W}_2(R(T, Y)Z, U, V, T) - g(U, V)Ric(R(T, Y)Z, T)$

$= \mathcal{R}(R(T, Y)Z, U, V, T) + \frac{1}{(n-1)}\{-g(R(T, Y)Z, V)Ric(U, T)\}$

$= A(R(T, Y)Z)g(U, V) - A(U)R(T, Y, Z, V)$

$+ (n-1)/(n-1)\{A(U)R(T, Y, Z, V) - g(U, V)A(R(T, Y)Z)\}$

$= A(R(T, Y)Z)g(U, V) - A(U)R(T, Y, Z, V)$

$+ A(U)R(T, Y, Z)V - g(U, V)A(R(T, Y)Z)$

$= 0$, \hspace{1cm} (2.9)

$\mathcal{W}_2(Z, U, R(T, Y)V, T)$

$= \mathcal{R}(Z, R(T, Y)V, T) + \frac{1}{(n-1)}\{g(Z, R(T, Y)V)Ric(U, T)\}$

$- g(U, V)Ric(R(T, Y)Z, T))$

$= A(R(T, Y)Z)g(U, V) - A(U)R(T, Y, V, Z)$

$+ \frac{n-1}{n-1}\{A(U)R(T, Y, V, Z) - A(Z)R(T, Y, V, U)\}$

$= 0$, \hspace{1cm} (2.10)

$W_2(Z, R(T, Y)U, T)$

$= \mathcal{W}_2(Z, T, V, T)g(Y, U) - A(U)\mathcal{W}_2(Z, Y, V, T)$

$= g(Y, U)\left\{\mathcal{R}(Z, T, V, T) + \frac{1}{n-1}\left[g(Z, V)Ric(T, T) - g(T, V)Ric(Z, T)\right]\right\}$

$- A(\mathcal{W}_2(Z, Y)V)\right\}$

$= g(Y, U)\{A(Z)A(V) - A(T)g(Z, V)\}$

$+ \frac{1}{n-1}\{-(n-1)g(V, Z) - (n-1)A(V)\} - A(\mathcal{W}_2(Z, Y)V)$
= g(Y, U)\{A(Z)A(V) + g(Z, V) - A(Z)A(V) - g(Z, V)\}

= 0 - 0

= 0, \quad (2.11)

since \(A(W_2(Z, Y)V) = 0\).

From (2.5) and on adding all the above terms, we have

\[-'W_2(Z, U, V, Y) = 0, \quad \Rightarrow 'W_2(Z, U, V, Y) = 0. \quad (2.12)\]

**Theorem 2.2.** The \(W_2\)-symmetric recurrent LP-Sasakian manifold is a manifold of constant scalar curvature and its curvature is given by \(r = n(n - 1)\).

**Proof.** Let \(\{e_i\}, \ i = 1, 2, ..., n\) be an orthonormal basis of the tangent space at any point. Then the sum for \(1 \leq i \leq n\) of the relation is given by

\['W_2(Z, U, V, Y) = 'R(Z, U, V, Y) + \frac{1}{(n - 1)}\{g(Z, V)Ric(U, V)Ric(Z, Y)\}.

When \(Z = Y = e_i\), then we have

\['W_2(e_i, U, V, e_i)

= 'R(e_i, U, V, e_i) + \frac{1}{n - 1}\{g(e_i, V)Ric(U, e_i) - g(U, V)Ric(e_i, e_i)\}.

Using (2.9), we have

\['R(e_i, U, V, e_i) = \frac{1}{n - 1}\{g(U, V)Ric(e_i, e_i) - g(e_i, V)Ric(U, e_i)\}

\['R(e_i, U, V, e_i) = \frac{1}{n - 1}\{2mg(e_i, e_i)g(U, V) - nRic(U, V)\}.

Now using the expansion for \('R\), we have

\['R(e_i, U, V, e_i) = 0, \quad \Rightarrow 0 = \frac{1}{n - 1}\{2nmg(U, V) - nRic(U, V)\}.

That is, \(Ric(U, V) = 2mg(U, V)\), which on contraction gives \(r = n(n - 1)\).
3. Symmetric and Skew-symmetric Parts of $W_2$-tensor

Let $P$ and $Q$ be the skew-symmetric and symmetric parts of $W_2$ tensor, with respect to $U$ and $Z$. These parts are defined as

$$P(X, Y, Z, U) = \frac{1}{2} \{ W_2(X, Y, Z, U) - 'W_2(X, Y, Z, U) \} = 'R(X, Y, Z, U)$$

$$+ \frac{1}{2(n-1)} \{ g(X, Z)Ric(Y, U) - g(Y, Z)Ric(X, U)$$

$$- g(X, U)Ric(Y, Z) + g(Y, U)Ric(Y, Z) \}, \tag{3.1}$$

$$Q(X, Y, Z, U) = \frac{1}{2} \{ W_2(X, Y, Z, U) + 'W_2(X, Y, U, Z) \} = \frac{1}{2(n-1)} \{ g(X, Z)Ric(Y, U) - g(Y, Z)Ric(X, U)$$

$$+ g(X, U)Ric(Y, Z) - g(Y, U)Ric(X, Z) \}. \tag{3.2}$$

**Theorem 3.1.** An LP-Sasakian manifold which is $P$-symmetric is necessarily a space of constant scalar curvature and its scalar curvature is $r = n(n-1)$.

**Proof.** If an LP-Sasakian manifold is $P$-symmetric, then

$$D_U P(X, Y) Z = P(X, Y, Z, U) = 0.$$ 

That is,

$$0 = 'R(X, Y, Z, U) + \frac{1}{2(n-1)} \{ g(X, Z)Ric(Y, U) - g(Y, Z)Ric(X, U) \}$$

$$= A(R(T, Y) Z) g(U, V) - A(U) R(T, Y, Z, V).$$

Now putting $X = U = e_i$, $i = 1, 2, ..., n$, we have

$$0 = 'R(e_i, Y, Z, e_i) + \frac{1}{2(n-1)} \{ g(e_i, Z)Ric(Y, e_i) - g(Y, Z)Ric(e_i, e_i)$$

$$- g(e_i, e_i)Ric(Y, Z) + g(Y, e_i)Ric(e_i, Z) \}. $$
That is,

\[ 0 = 'R(e_i, Y, Z, e_i) \]

\[ + \frac{1}{2(n-1)} \{ n \text{Ric}(Y, Z) - 2nmg(Y, Z) - n \text{Ric}(Y, Z) + n \text{Ric}(Y, Z) \} \]

or

\[ 'R(e_i, Y, Z, e_i) = \frac{-1}{2(n-1)} \{ n \text{Ric}(Y, Z) - 2nmg(Y, Z) \}. \]

But

\[ 'R(e_i, Y, Z, e_i) = g(e_i, Y)g(Z, e_i) - g(Y, Z)g(e_i, e_i) \]

\[ = g(Y, Z) - g(Y, Z) = 0. \]

Thus from the above, we have

\[ 0 = \frac{-1}{2(n-1)} \{ n \text{Ric}(Y, Z) - 2nmg(Y, Z) \} \Rightarrow \text{Ric}(Y, Z) = 2mg(Y, Z), \]

which again on transvecting, gives \( r = 2nm = n(n-1), \) where \( n = 2m + 1. \)

**Theorem 3.2.** An LP-Sasakian manifold which is Q-symmetric is a space of constant scalar curvature and its scalar curvature is \( r = n(n-1). \)

**Proof.** If an LP-Sasakian manifold is Q-symmetric, then we have

\[ D_U Q(X, Y)Z = Q(X, Y, Z, U) \]

\[ = 0 \]

\[ = \frac{1}{2(n-1)} \{ g(X, Z) \text{Ric}(Y, U) - g(Y, Z) \text{Ric}(X, U) \]

\[ + g(X, U) \text{Ric}(Y, Z) - g(Y, U) \text{Ric}(X, Z) \}. \]

Putting \( X = U = e_i, i = 1, 2, ..., n \) in the above equation, we have
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$$0 = \frac{1}{2(n-1)} \{ g(e_i, Z)Ric(Y, e_i) - g(Y, Z)Ric(e_i, e_i) \\
+ g(e_i, e_i)Ric(Y, Z) - g(Y, e_i)Ric(e_i, Z) \}$$

$$= \frac{1}{2(n-1)} \{ nRic(Y, Z) - 2nmg(Y, Z) + nRic(Y, Z) - nRic(Y, Z) \}$$

or

$$nRic(Y, Z) - 2nmg(Y, Z) = 0 \Rightarrow Ric(Y, Z) = 2mg(Y, Z).$$

Transvecting this Ricci tensor, we have $r = n(n-1) = 2nm$, where $n = 2m + 1$.

**Remark.** The contracted part of the skew-symmetric part of $W_2$ vanishes identically and we use this to extend the Pirani formulation of gravitational waves to an Einstein space like $W_{12}$ (Pokhariyal [6]). The non-vanishing of the divergence of complexion vector shows that we cannot use this tensor to reduce the electromagnetic field to purely electric field.

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**References**


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