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A NOTE ON METRIC EQUIVALENCE OF SOME OPERATORS

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Abstract

In this paper, we study the metric equivalence relation and closely related relations on some classes of operators. We describe the spectral picture of metrically equivalent operators. We give some conditions when metric equivalence of operators implies unitary equivalence of operators.

1. Introduction

Let $\mathcal{H}$ denote a Hilbert space and $B(\mathcal{H})$ denote the Banach algebra of bounded linear operators. If $T \in B(\mathcal{H})$, then T^* denotes the adjoint of T, while $Ker(T)$, $Ran(T)$, $\overline{\mathcal{M}}$ and $\mathcal{M}^{\perp}$ stands for the kernel of T, range of T,

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closure of $\mathcal{M}$ and orthogonal complement of a closed subspace $\mathcal{M}$ of $\mathcal{H}$, respectively. We denote by $\sigma(T)$, $\| T \|$ and $W(T)$, the spectrum, norm and numerical range of T, respectively. Recall that an operator $T \in B(\mathcal{H})$ is

normal if $T^*T = TT^*$.

self-adjoint (*or hermitian*) if $T^* = T$.

skew-adjoint if $T^* = -T$.

unitary if $T^*T = TT^* = I$.

quasinormal if $T(T^*T) = (T^*T)T$.

binormal if $(T^*T)(TT^*) = (TT^*)(T^*T)$.

a *projection* if $T^2 = T$ *and* $T^* = T$.

an *involution* if $T^2 = I$.

a *symmetry* if $T = T^* = T^{-1}$. That is, T is a self-adjoint unitary.

isometric if $T^*T = I$.

Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are said to be *similar* (denoted by $A \sim B$) if there exists an invertible operator $N \in B(\mathcal{H}, \mathcal{K})$ such that $NA = BN$ or equivalently $A = N^{-1}BN$, and are *unitarily equivalent* (denoted by $A \cong B$) if there exists a unitary operator $U \in B_+(\mathcal{H}, \mathcal{K})$ (Banach algebra of all invertible operators in $B(\mathcal{H})$) such that $UA = BU$ (i.e., $A = U^*BU$, equivalently, $A = U^{-1}BU$). Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are said to be *metrically equivalent* (denoted by $A \sim_m B$) if $\| Ax \| = \| Bx \|$, (equivalently, $|\langle Ax, Ax\rangle|^{1/2} = |\langle Bx, Bx\rangle|^{1/2}$ for all $x \in \mathcal{H}$). Clearly similarity, unitary equivalence and metric equivalence are equivalence relations on $B(\mathcal{H})$.

2. Main Results

Theorem 2.1. *If T is a normal operator and* $S \in B(\mathcal{H})$ *is unitarily equivalent to T, then S is normal.*

Proof. Suppose $S = U^*TU$, where U is unitary and T is normal. Then

$$\begin{aligned} S^*S &= (U^*T^*U)(U^*TU) \\ &= U^*T^*TU \\ &= U^*TT^*U \\ &= SU^*T^*U \\ &= SU^*US^* \\ &= SS^* \end{aligned}$$

which proves the claim.

Theorem 2.2 ([4, Theorem 3]). *A necessary and sufficient condition that an operator* $T \in B(\mathcal{H})$ *be normal is that* $\| Tx \| = \| T^*x \|$ *for every* $x \in \mathcal{H}$.

Corollary 2.3. *An operator* $T \in B(\mathcal{H})$ *is normal if and only if T and* T^* *are metrically equivalent.*

Proof. The proof follows easily from Theorem 2.1.

Theorem 2.4. *If T is a normal operator, then there exists a unitary operator U such that* $T^* = UT$.

Theorem 2.5 ([5, Theorem 2, p. 56]). *Let S and T be bounded linear operators on a Hilbert space* $\mathcal{H}$. *If* $T^*T = S^*S$, *then there exists a partial isometry U such that the initial space* $\mathcal{M} = \overline{Ran(T)}$ *and the final space* $\mathcal{N} = \overline{Ran(S)}$, *and* $S = UT$.

Corollary 2.6. *If S and T are metrically equivalent normal operators, then there exists a unitary operator U such that* $S = UT$.