Lesson 6

• **Magnets**
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  - Torque on a Current Loop in a Uniform Magnetic Field
  - Motion of a Charged Particle in a Uniform Magnetic Field
• **Applications Involving Charged Particles Moving in a Magnetic Field**
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Magnetic Fields and Forces

We can define a magnetic field $\mathbf{B}$ at some point in space in terms of the magnetic force $\mathbf{F}_B$ that the field exerts on a charged particle moving with a velocity $\mathbf{v}$, which we call the test object. For the time being, let us assume that no electric or gravitational fields are present at the location of the test object. Experiments on various charged particles moving in a magnetic field give the following results:

- The magnitude $F_B$ of the magnetic force exerted on the particle is proportional to the charge $q$ and to the speed $v$ of the particle.
- The magnitude and direction of $\mathbf{F}_B$ depend on the velocity of the particle and on the magnitude and direction of the magnetic field $\mathbf{B}$.
- When a charged particle moves parallel to the magnetic field vector, the magnetic force acting on the particle is zero.

$$\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}$$
Magnetic Fields and Forces

When the particle’s velocity vector makes any angle $\theta \neq 0$ with the magnetic field, the magnetic force acts in a direction perpendicular to both $\mathbf{v}$ and $\mathbf{B}$; that is, $\mathbf{F}_B$ is perpendicular to the plane formed by $\mathbf{v}$ and $\mathbf{B}$.

\[ \mathbf{F}_B = q\mathbf{v} \times \mathbf{B} \]

The magnitude of the magnetic force on a charged particle is

\[ F_B = |q|vB \sin \theta \]

where $\theta$ is the smaller angle between $\mathbf{v}$ and $\mathbf{B}$. The SI unit of $\mathbf{B}$ is the tesla (T), where $1 \text{T} = 1 \text{N/A} \cdot \text{m}$.

If a straight conductor of length $L$ carries a current $I$, the force exerted on that conductor when it is placed in a uniform magnetic field $\mathbf{B}$ is

\[ \mathbf{F}_B = I/L \times \mathbf{B} \]
Magnetic Fields and Forces

The magnetic force exerted on a positive charge is in the direction opposite the direction of the magnetic force exerted on a negative charge moving in the same direction. The magnitude of the magnetic force exerted on the moving particle is proportional to $\sin \theta$, where $\theta$ is the angle the particle’s velocity vector makes with the direction of $\mathbf{B}$. We can summarize these observations by writing the magnetic force in the form

When a charged particle moves in a magnetic field, the work done by the magnetic force on the particle is zero because the displacement is always perpendicular to the direction of the force. The magnetic field can alter the direction of the particle’s velocity vector, but it cannot change its speed.
Magnetic Fields and Forces

Two right-hand rules for determining the direction of the magnetic force $F_B = qv \times B$ acting on a particle with charge $q$ moving with a velocity $v$ in a magnetic field $B$. (a) In this rule, the fingers point in the direction of $v$, with $B$ coming out of your palm, so that you can curl your fingers in the direction of $B$. The direction of $v \times B$, and the force on a positive charge, is the direction in which the thumb points. (b) In this rule, the vector $v$ is in the direction of your thumb and $B$ in the direction of your fingers. The force $F_B$ on a positive charge is in the direction of your palm, as if you are pushing the particle with your hand.
Magnetic Fields and Forces

There are several important differences between electric and magnetic forces:

• The electric force acts along the direction of the electric field, whereas the magnetic force acts perpendicular to the magnetic field.

• The electric force acts on a charged particle regardless of whether the particle is moving, whereas the magnetic force acts on a charged particle only when the particle is in motion.

• The electric force does work in displacing a charged particle, whereas the magnetic force associated with a steady magnetic field does no work when a particle is displaced because the force is perpendicular to the displacement.

The **SI unit** of magnetic field is the newton per coulomb-meter per second, which is called the **tesla (T)**.
Magnetic Fields and Forces

• Example: An Electron Moving in a Magnetic Field

An electron in a television picture tube moves toward the front of the tube with a speed of \(8.0 \times 10^6\) m/s along the x axis. Surrounding the neck of the tube are coils of wire that create a magnetic field of magnitude 0.025 T, directed at an angle of 60° to the x axis and lying in the xy plane.

(i) Calculate the magnetic force on the electron

(ii) Find a vector expression for the magnetic force on the electron

Solution

\[ F_B = |q|vB \sin \theta \]

\[ = (1.6 \times 10^{-19} \text{ C}) (8.0 \times 10^6 \text{ m/s}) (0.025 \text{ T}) \sin 60^\circ \]

\[ = 2.8 \times 10^{-14} \text{ N} \]
Magnetic Fields and Forces

(ii)

\[ \mathbf{v} = (8.0 \times 10^6 \hat{i}) \text{ m/s} \]

and one for the magnetic field:

\[ \mathbf{B} = (0.025 \cos 60^\circ \hat{i} + 0.025 \sin 60^\circ \hat{j}) \text{T} \]
\[ = (0.013 \hat{i} + 0.022 \hat{j}) \text{T} \]

\[ \mathbf{F}_B = q \mathbf{v} \times \mathbf{B} \]
\[ = (-e)[(8.0 \times 10^6 \hat{i}) \text{ m/s}] \times [(0.013 \hat{i} + 0.022 \hat{j}) \text{T}] \]
\[ = (-e)[(8.0 \times 10^6 \hat{i}) \text{ m/s}] \times [(0.013 \hat{i}) \text{T}] \]
\[ + (-e)[(8.0 \times 10^6 \hat{i}) \text{ m/s}] \times [(0.022 \hat{j}) \text{T}] \]
\[ = (-e)(8.0 \times 10^6 \text{ m/s})(0.013 \text{ T})(\hat{i} \times \hat{i}) \]
\[ + (-e)(8.0 \times 10^6 \text{ m/s})(0.022 \text{ T})(\hat{i} \times \hat{j}) \]
\[ = (-1.6 \times 10^{-19} \text{ C})(8.0 \times 10^6 \text{ m/s})(0.022 \text{ T}) \hat{k} \]
\[ \mathbf{F}_B = (-2.8 \times 10^{-14} \text{ N}) \hat{k} \]

Unit vectors \( \hat{i}, \hat{j}, \text{ and } \hat{k} \) obey the following rules:

\[ \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \]
\[ \hat{i} \times \hat{j} = -\hat{j} \times \hat{i} = \hat{k} \]
\[ \hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i} \]
\[ \hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j} \]
Magnetic Force Acting on a Current-Carrying Conductor

- If a magnetic force is exerted on a single charged particle when the particle moves through a magnetic field, a current-carrying wire also experiences a force when placed in a magnetic field.
- This follows from the fact that the current is a collection of many charged particles in motion; hence, the resultant force exerted by the field on the wire is the vector sum of the individual forces exerted on all the charged particles making up the current.
- The force exerted on the particles is transmitted to the wire when the particles collide with the atoms making up the wire.
A segment of a current-carrying wire in a magnetic field $B$. The magnetic force exerted on each charge making up the current is $q\mathbf{v}_d \times \mathbf{B}$ and the net force on the segment of length $L$ is $I\mathbf{L} \times \mathbf{B}$. 

(a) Magnetic field lines coming out of the paper are indicated by dots, representing the tips of arrows coming outward.

(b) Magnetic field lines going into the paper are indicated by crosses, representing the feathers of arrows going inward.
Magnetic Force Acting on a Current-Carrying Conductor

Now consider an arbitrarily shaped wire segment of uniform cross section in a magnetic field, as shown in the Figure. It follows the magnetic force exerted on a small segment of vector length $ds$ in the presence of a field $\mathbf{B}$ is

$$d\mathbf{F}_B = I\, ds \times \mathbf{B}$$

(a) A curved wire carrying a current $I$ in a uniform magnetic field. The total magnetic force acting on the wire is equivalent to the force on a straight wire of length $L'$ running between the ends of the curved wire.

(b) A current-carrying loop of arbitrary shape in a uniform magnetic field. The net magnetic force on the loop is zero.
Magnetic Force Acting on a Current-Carrying Conductor

**Case 1.** A curved wire carries a current $I$ and is located in a uniform magnetic field $\mathbf{B}$,

$$
\mathbf{F}_B = I \int_a^b d\mathbf{s} \times \mathbf{B}
$$

where $a$ and $b$ represent the end points of the wire.

$$
\mathbf{F}_B = I \left( \int_a^b d\mathbf{s} \right) \times \mathbf{B}
$$

But the quantity $\int_a^b d\mathbf{s}$ represents the *vector sum* of all the length elements from $a$ to $b$. From the law of vector addition, the sum equals the vector $\mathbf{L}'$, directed from $a$ to $b$.

Therefore,

$$
\mathbf{F}_B = I \mathbf{L}' \times \mathbf{B}
$$
**Magnetic Force Acting on a Current-Carrying Conductor**

**Case 2.** An arbitrarily shaped closed loop carrying a current $I$ is placed in a uniform magnetic field, as shown in Figure b. We can again express the magnetic force

$$
\mathbf{F}_B = I \left( \oint ds \right) \times \mathbf{B}
$$

Because the set of length elements forms a closed polygon, the vector sum must be zero. This follows from the procedure for adding vectors by the graphical method. Because $\oint ds = 0$, we conclude that $\mathbf{F}_B = 0$; that is, the net magnetic force acting on any closed current loop in a uniform magnetic field is zero.
**Example: Force on a Semicircular Conductor**

A wire bent into a semicircle of radius $R$ forms a closed circuit and carries a current $I$. The wire lies in the $xy$ plane, and a uniform magnetic field is directed along the positive $y$ axis. Find the magnitude and direction of the magnetic force acting on the straight portion of the wire and on the curved portion.

**Solution**  The magnetic force $\mathbf{F}_1$ acting on the straight portion has a magnitude $F_1 = ILB = 2IRB$ because $L = 2R$ and the wire is oriented perpendicular to $\mathbf{B}$. The direction of $\mathbf{F}_1$ is out of the page based on the right-hand rule for the cross product $\mathbf{L} \times \mathbf{B}$.

Because the wire lies in the $xy$ plane, the two forces on the loop can be expressed as

\[
\mathbf{F}_1 = 2IRB \hat{k}
\]

\[
\mathbf{F}_2 = -2IRB \hat{k}
\]

The net magnetic force on the loop is

\[
\sum \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = 2IRB \hat{k} - 2IRB \hat{k} = 0
\]
Torque on a Current Loop in a Uniform Magnetic Field

Consider a rectangular loop carrying a current $I$ in the presence of a uniform magnetic field directed parallel to the plane of the loop, as shown in Figure a. No magnetic forces act on sides 1 and 3 because these wires are parallel to the field; hence, $L \times B = 0$ for these sides. However, magnetic forces do act on sides 2 and 4 because these sides are oriented perpendicular to the field. The magnitude of these forces is

$$ F_2 = F_4 = IaB $$

The direction of $F_2$, the magnetic force exerted on wire 2, is out of the page in the view, and that of $F_4$, the magnetic force exerted on wire 4, is into the page in the same view. If we view the loop from side 3 and sight along sides 2 and 4, we see the view shown in Figure b, and the two magnetic forces $F_2$ and $F_4$ are directed as shown. Note that the two forces point in opposite directions but are not directed along the same line of action.
Torque on a Current Loop in a Uniform Magnetic Field

If the loop is pivoted so that it can rotate about point O, these two forces produce about O a torque that rotates the loop clockwise. The magnitude of this torque $\tau_{\text{max}}$ is

$$\tau_{\text{max}} = F_2 \frac{b}{2} + F_4 \frac{b}{2} = (IaB) \frac{b}{2} + (IaB) \frac{b}{2} = IabB$$

where the moment arm about $O$ is $b/2$ for each force. Because the area enclosed by the loop is $A = ab$, we can express the maximum torque as

$$\tau_{\text{max}} = IAB$$
Torque on a Current Loop in a Uniform Magnetic Field

Now suppose that the uniform magnetic field makes an angle $\theta < 90^\circ$ with a line perpendicular to the plane of the loop, as in the Figure. For convenience, we assume that $\mathbf{B}$ is perpendicular to sides 2 and 4. In this case, the magnetic forces $\mathbf{F}_1$ and $\mathbf{F}_3$ exerted on sides 1 and 3 cancel each other and produce no torque because they pass through a common origin. However, the magnetic forces $\mathbf{F}_2$ and $\mathbf{F}_4$ acting on sides 2 and 4 produce a torque about any point.

Referring to the end view, we note that the moment arm of $\mathbf{F}_2$ about the point O is equal to $(b/2) \sin \theta$. Likewise, the moment arm of $\mathbf{F}_4$ about O is also $(b/2) \sin \theta$. Because $\mathbf{F}_2 = \mathbf{F}_4 = Ia\mathbf{B}$, the magnitude of the net torque about O is

$$
\tau = F_2 \frac{b}{2} \sin \theta + F_4 \frac{b}{2} \sin \theta
$$

$$
= IaB \left( \frac{b}{2} \sin \theta \right) + IaB \left( \frac{b}{2} \sin \theta \right) = IabB \sin \theta
$$

$$
= IAB \sin \theta
$$

where $A = ab$ is the area of the loop.
Torque on a Current Loop in a Uniform Magnetic Field

A convenient expression for the torque exerted on a loop placed in a uniform magnetic field \( \mathbf{B} \) is

\[
\tau = I \mathbf{A} \times \mathbf{B}
\]

The product \( I \mathbf{A} \) is defined to be the magnetic dipole moment \( \mu \) (often simply called the “magnetic moment”) of the loop:

\[
\mu = I \mathbf{A}
\]

The SI unit of magnetic dipole moment is ampere-meter\(^2\) (A \cdot m\(^2\)). Using this definition, we can express the torque exerted on a current-carrying loop in a magnetic field \( \mathbf{B} \) as

\[
\tau = \mu \times \mathbf{B}
\]
Torque on a Current Loop in a Uniform Magnetic Field

• **Example: The Magnetic Dipole Moment of a Coil**

A rectangular coil of dimensions $5.40 \text{ cm} \times 8.50 \text{ cm}$ consists of 25 turns of wire and carries a current of $15.0 \text{ mA}$. A $0.350\text{-T}$ magnetic field is applied parallel to the plane of the loop.

**(A)** Calculate the magnitude of its magnetic dipole moment.

**Solution**

$$\mu_{\text{coil}} = NIA = (25)(15.0 \times 10^{-3} \text{ A})(0.0540 \text{ m})(0.0850 \text{ m})$$

$$= 1.72 \times 10^{-3} \text{ A} \cdot \text{m}^2$$

**(B)** What is the magnitude of the torque acting on the loop?

**Solution**

$$\tau = \mu_{\text{coil}}B = (1.72 \times 10^{-3} \text{ A} \cdot \text{m}^2)(0.350 \text{ T})$$

$$= 6.02 \times 10^{-4} \text{ N} \cdot \text{m}$$
When the velocity of a charged particle is perpendicular to a uniform magnetic field, the particle moves in a circular path in a plane perpendicular to $B$. The magnetic force $F_B$ acting on the charge is always directed toward the center of the circle.

\[ \sum F = ma_c \]

\[ \omega = \frac{v}{r} = \frac{qB}{m} \]

\[ F_B = qvB = \frac{mv^2}{r} \]

\[ r = \frac{mv}{qB} \]

\[ T = \frac{2\pi r}{v} = \frac{2\pi}{\omega} = \frac{2\pi m}{qB} \]

These results show that the angular speed of the particle and the period of the circular motion do not depend on the linear speed of the particle or on the radius of the orbit. The angular speed $\omega$ is often referred to as the **cyclotron frequency** because charged particles circulate at this angular frequency in the type of accelerator called a **cyclotron**.
Motion of a Charged Particle in a Uniform Magnetic Field

If a charged particle moves in a uniform magnetic field with its velocity at some arbitrary angle with respect to \( B \), its path is a **helix**. For example, if the field is directed in the \( x \) direction, as shown in the Figure, there is no component of force in the \( x \) direction. As a result, \( a_x = 0 \), and the \( x \) component of velocity remains constant.

However, the magnetic force \( q \mathbf{v} \times \mathbf{B} \) causes the components \( v_y \) and \( v_z \) to change in time, and the resulting motion is a helix whose axis is parallel to the magnetic field. The projection of the path onto the \( yz \) plane (viewed along the \( x \) axis) is a circle. (The projections of the path onto the \( xy \) and \( xz \) planes are sinusoids!)

\[
v \text{ is replaced by } \quad v_\perp = \sqrt{v_y^2 + v_z^2}.
\]
Motion of a Charged Particle in a Uniform Magnetic Field

• Example: A Proton Moving Perpendicular to a Uniform Magnetic Field

A proton is moving in a circular orbit of radius 14 cm in a uniform 0.35-T magnetic field perpendicular to the velocity of the proton. Find the linear speed of the proton.

Solution

\[ v = \frac{qBr}{m_p} = \frac{(1.60 \times 10^{-19} \text{ C})(0.35 \text{ T})(0.14 \text{ m})}{1.67 \times 10^{-27} \text{ kg}} \]

\[ = 4.7 \times 10^6 \text{ m/s} \]
Motion of a Charged Particle in a Uniform Magnetic Field

• Example: Bending an Electron Beam

In an experiment designed to measure the magnitude of a uniform magnetic field, electrons are accelerated from rest through a potential difference of 350 V. The electrons travel along a curved path because of the magnetic force exerted on them, and the radius of the path is measured to be 7.5 cm. If the magnetic field is perpendicular to the beam,

(i) what is the magnitude of the field?

(ii) What is the angular speed of the electrons?

Solution

(i) 

\[
\Delta K + \Delta U = 0 \rightarrow \frac{1}{2}m_e v^2 + (-e) \Delta V = 0
\]

\[
v = \sqrt{\frac{2e \Delta V}{m_e}} = \sqrt{\frac{2(1.60 \times 10^{-19} \text{ C})(350 \text{ V})}{9.11 \times 10^{-31} \text{ kg}}}
\]

\[= 1.11 \times 10^7 \text{ m/s}\]
Motion of a Charged Particle in a Uniform Magnetic Field

\[ B = \frac{m_e v}{er} = \frac{(9.11 \times 10^{-31}\text{ kg}) (1.11 \times 10^7\text{ m/s})}{(1.60 \times 10^{-19}\text{ C}) (0.075\text{ m})} = 8.4 \times 10^{-4}\text{ T} \]

(ii)

\[ \omega = \frac{v}{r} = \frac{1.11 \times 10^7\text{ m/s}}{0.075\text{ m}} = 1.5 \times 10^8\text{ rad/s} \]
Applications Involving Charged Particles Moving in a Magnetic Field

A charge moving with a velocity $\mathbf{v}$ in the presence of both an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ experiences both an electric force $q\mathbf{E}$ and a magnetic force $q\mathbf{v} \times \mathbf{B}$. The total force (called the Lorentz force) acting on the charge is

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

(1) **Velocity Selector**

(a) A velocity selector. When a positively charged particle is moving with velocity $\mathbf{v}$ in the presence of a magnetic field directed into the page and an electric field directed downward, it experiences a downward electric force $q\mathbf{E}$ and an upward magnetic force $q \mathbf{v} \times \mathbf{B}$.

(b) When these forces balance, the particle moves in a horizontal line through the fields.
Applications Involving Charged Particles Moving in a Magnetic Field

(2) **The Mass Spectrometer**

A mass spectrometer separates ions according to their mass-to-charge ratio. In one version of this device, known as the Bainbridge mass spectrometer, a beam of ions first passes through a velocity selector and then enters a second uniform magnetic field $B_0$ that has the same direction as the magnetic field in the selector.

Positively charged particles are sent first through a velocity selector and then into a region where the magnetic field $B_0$ causes the particles to move in a semicircular path and strike a detector array.

(3) **The Cyclotron**

A cyclotron is a device that can accelerate charged particles to very high speeds. The energetic particles produced are used to bombard atomic nuclei and thereby produce nuclear reactions of interest to researchers. A number of hospitals use cyclotron facilities to produce radioactive substances for diagnosis and treatment.
The Hall Effect

• When a current-carrying conductor is placed in a magnetic field, a potential difference is generated in a direction perpendicular to both the current and the magnetic field.

• This phenomenon, first observed by Edwin Hall (1855–1938) in 1879, is known as the Hall effect. It arises from the deflection of charge carriers to one side of the conductor as a result of the magnetic force they experience. The Hall effect gives information regarding the sign of the charge carriers and their density; it can also be used to measure the magnitude of magnetic fields.
The Biot–Savart Law

The magnetic field \( dB \) at a point due to the current \( I \) through a length element \( ds \) is given by the Biot–Savart law. The direction of the field is out of the page at \( P \) and into the page at \( P' \).

- The vector \( dB \) is perpendicular both to \( ds \) (which points in the direction of the current) and to the unit vector \( \hat{r} \) directed from \( ds \) toward \( P \).
- The magnitude of \( dB \) is inversely proportional to \( r^2 \), where \( r \) is the distance from \( ds \) to \( P \).
- The magnitude of \( dB \) is proportional to the current and to the magnitude \( ds \) of the length element \( ds \).
- The magnitude of \( dB \) is proportional to \( \sin \theta \), where \( \theta \) is the angle between the vectors \( ds \) and \( \hat{r} \).

These observations are summarized in the mathematical expression known today as the Biot–Savart law:

\[
dB = \frac{\mu_0}{4\pi} \frac{I ds \times \hat{r}}{r^2}
\]

where \( \mu_0 \) is a constant called the permeability of free space:

\[
\mu_0 = 4\pi \times 10^{-7} \text{ T} \cdot \text{m/A}
\]